Research on the Combinatorial Transform Mathematics Problem "Frog Leap"

High school affiliated to Renmin university of China   Zengtian

Abstract

In this paper we extend and deepen a shortlist for the 37th International Mathematical Olympiad (IMO) [1] and propose the Frog Leap Commute Theorem and the Queue Polynomial. We explore the problem from the following aspects:

(1) Make use of semi-invariants and propose the Frog Leap Commute Theorem.
(2) Make extensions regarding frogs leaping to opposite directions on a straight line.
(3) Research frogs leaping to the same direction on a straight line and solve the minimum number of frogs satisfying an infinite leap.
(4) Extend the problem to leaps on a plane or in space.
(5) Research and extend problems regarding frogs leaping on a circle.
(6) Estimate the function $c(n)$ and calculate the order of the function.

Key words

Frog Leap   Leap Commute Theorem   Queue Polynomial   Positive State

1. Original Problem

The original problem in *The 37th IMO Shortlist* is as follows:

A finite number of beans are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one bean is chosen. Two beans are taken from this square; one of them is placed on the square immediately to the left, and the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one bean on each square. Given some initial configuration, it shows that any legal sequence of moves will terminate after the same number of steps and with the same final configuration [2].

For the convenience of description and further extension, we change the original problem above into the following one: A finite number of frogs are on a straight line. If two frogs are at the same point, one of them will leap one unit to the left and the other one unit to the right. Will the frogs leap infinitely or terminate after some sequence of steps?

Suppose there are n frogs and their coordinates are $x_1, x_2, ..., x_n (x_i \in Z)$ respectively. When two frogs on coordinate x leap, one coordinate changes into $x+1$, while the other into $x-1$.

We consider a semi-invariant (a variable that monotonically increases or decreases during the whole leaping progress) $S = {x_1}^2 + {x_2}^2 + ... + {x_n}^2$. Since $(x+1)^2 + (x-1)^2 - 2x^2 = 2$, S increases by 2 for each leap. If the frogs can leap infinitely, S increases continuously.

Consider two adjacent points. If there were frogs on them, there will always be frogs on