On the upper bound of number-theoretic function $F_f(h)$

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Abstract: The problem of the existence of infinitely many prime values of a number-theoretic function $f(x)$ has been one of the most important topics in Number Theory. Note that if $f(x)$ represents infinitely many primes, then we can get this necessary condition: for any positive integer $h$, there exists a positive integer $k$ such that $(f(k), h) = 1$ and $f(k) > 1$. Naturally, we are interested in the number-theoretic functions $f(x)$ that satisfy the aforementioned necessary condition. Thus, there must exist the least positive integer $n$ such that $(f(n), h) = 1$ and $f(n) > 1$. Denote this least positive integer $n$ by $F_f(h)$. In this paper, we mainly focus on three famous number-theoretic functions: $f(x) = 2^x + 1$, $m(x) = 2^x - 1$ and $l(x) = x^2 + 1$, proving they satisfy the aforementioned necessary condition respectively. Furthermore, we approximately estimate the upper bound of $F_{f(x)}(h), F_{m(x)}(h), F_{l(x)}(h)$ respectively, and obtain some interesting results.

1. Introduction

Let $f(x)$ be a number-theoretic function. Whether $f(x)$ represents infinitely many primes has always been a problem that attracts great interests among many famous mathematicians. As early as 2000 years ago, Euclid has proved that $f(x) = x$ represents infinitely many primes. In 1837, Dirichlet proved that $f(x) = ax + b$ also represents infinitely many prime values, where $a$ and $b$ are integers with $(a, b) = 1$, either $a > 0, b \neq 0$ or $a = 1, b = 0$. By proving this, he completely solved this problem in the case of linear polynomial with integral coefficients. Apart from this case of linear polynomial, however, the problem becomes complex and there is still no complete solution by now. For example, whether functions such as $f(x) = 2^x + 1, m(x) = 2^x - 1$ and $l(x) = x^2 + 1$ represent infinitely many primes has not been verified until today. Actually, these functions are respectively related to the problem of Fermat number, Mersenne number and the first conjecture of Landau. Readers can refer to [6] for some other information. [6] is a paper that summaries the research history from Euclid to Green-Tao Theorem, studies the infinitude of some special kinds of prime and brings up some interesting questions. Our paper is just based on one of these questions. I thank Doctor Shaohua Zhang here for suggesting this interesting topic to us.

Note that if a number-theoretic function $f(x)$ represents infinitely many primes, we can get the